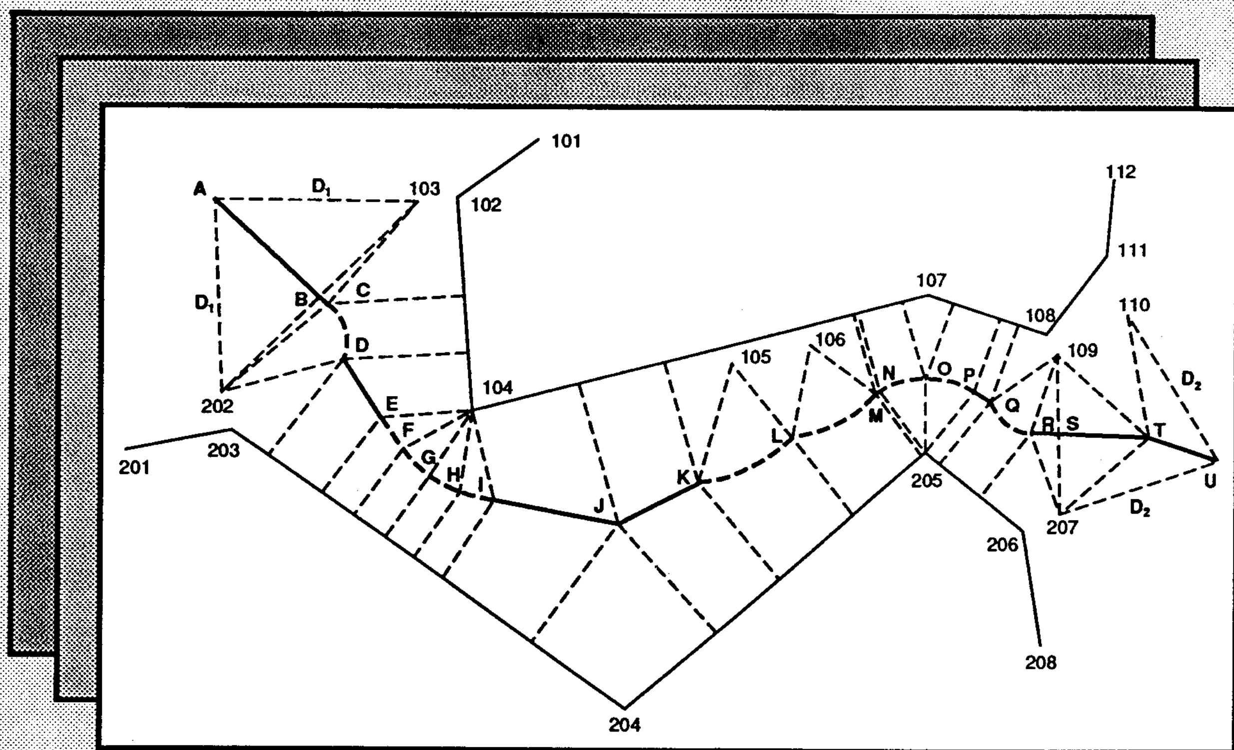
Geometric Median Line Computations



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GEOMETRIC MEDIAN LINE COMPUTATIONS

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ABSTRACT

The geometric median line of a body of water is defined as a series of intersecting line segments every point of which is equidistant from the nearest point on opposite shorelines.

Analytical procedures are presented that can be used to compute the exact coordinates of the points of intersection of all line segments that form the median line. An accurate legal description of the line can then be prepared in terms of those coordinates.

INTRODUCTION

The geometric median line of a body of water is a line every point of which is equidistant from the nearest point on opposite shorelines. It is continuous in the sense that it contains no gaps, and is formed by a series of intersecting line segments.

Two different types of median lines are described following: median lines based upon shorelines defined by salient points only, and median lines based upon shorelines defined by meander traverses. In the case of shorelines defined by salient points the median line consists of a series of intersecting straight line segments only. When shorelines are defined by meander lines the median line is formed by intersecting straight line segments and curved line segments.

The procedures outlined herein for locating median lines might be described as semi-graphical or semi-analytical. The approximate location of all median line turning points are determined graphically, the geometric shoreline condition that causes the median line direction change is identified, and the exact coordinates of median line turning points are then computed analytically using equations presented in the Appendices.* A computer program has been written by the author which can also be used to compute the coordinates of turning points after the geometric shoreline condition has been identified.

The computations described are subject to the same limitations as any other plane survey method.

SHORELINES DEFINED BY SALIENT POINTS

If a shoreline is described only by coordinates of a finite number of selected, prominently located points along the water's edge, it is said

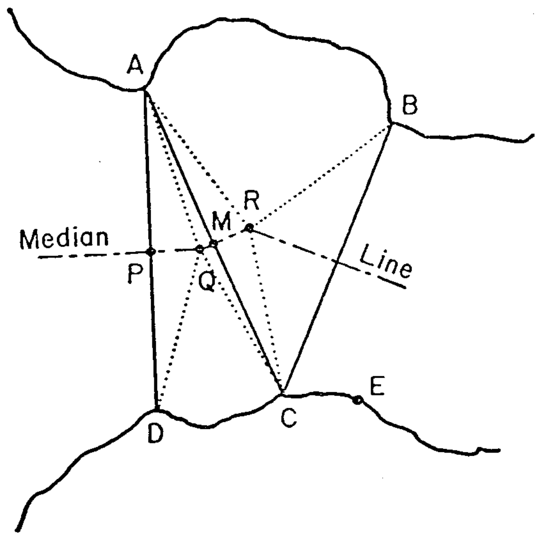


FIG. 1 – SHORELINE DEFINED BY SALIENT POINTS

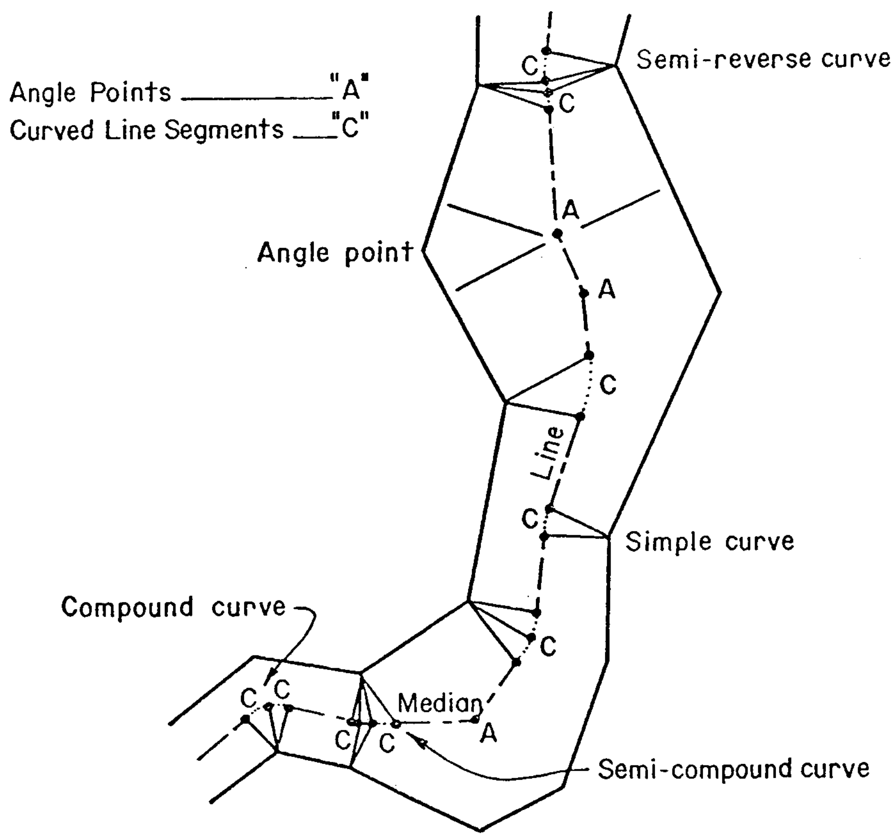


FIG. 2 – SHORELINE DEFINED BY MEANDER LINES

to be a shoreline defined by salient points. In such cases the geometric median line consists of a series of intersecting straight line segments only. A median line of this type is shown in Fig. 1.

Straight line segments of the median line, such as segment QR in Fig. 1, represent a portion of the line every point of which is equidistant from two salient points only, point A and point C, one on each opposite shoreline. The median line passes through a point M midway between points A and C and is perpendicular to line AC joining the two points. Median line angle points occur at the intersection of adjacent straight line segments. At angle points, such as points Q and R in Fig. 1, the median line is equidistant from three (or more on rare occasions) salient points,at least one of which is located on the opposite shoreline from the other two points.

DETERMINATION OF THE LOCATION OF ANGLE POINTS

The approximate locations of Angle Points are determined quite easily by graphical means. They can be located either by extending straight line segments of the median line until they intersect, or by locating the center points of circles that pass through three (or more) salient points. The coordinates of the exact positions of angle points are then computed analytically using equations presented in Appendix VIII.

COMPUTATION OF THE COORDINATES OF ANGLE POINTS

The plane coordinates Xr, Yr of a median line angle point, such as point R in Fig. 1, which is equidistant from the three salient points A, B and C having known coordinates Xa, Ya, Xb, Yb, and Xc, Yc can be determined by application of Eqs. 1-5 of Appendix VIII.

The coordinates of a point on the median line exactly halfway between two salient points, such as point P midway between points A and D, are given by Eqs. 6 and 7 of Appendix VIII.

SHORELINES DEFINED BY MEANDER LINES

If the shorelines of a body of water are defined by meander lines, the median line will consist of a continuous series of intersecting straight line segments and curved line segments. A median line of this type is shown in Fig. 2.

Median line direction changes occur either abruptly as angular deflections at the point of intersection of straight line segments, or gradually along curved line segments.

Straight line segments exist where all points of the median line are equidistant from two straight lines on opposite shorelines; or under certain conditions, when the median line passes between two meander line angle points that are convex toward the median line and located on opposite shorelines.

Curved line segments are generally parabolic, all points of which are equidistant from a meander line angle point on one shoreline (which is convex toward the median line) and a straight meander line segment on the opposite shoreline. Median line curves can be simple parabolic curves consisting of independent curved line segments separated by straight line segments, or can be relatively complex curves consisting of a number of intersecting curved line segments.

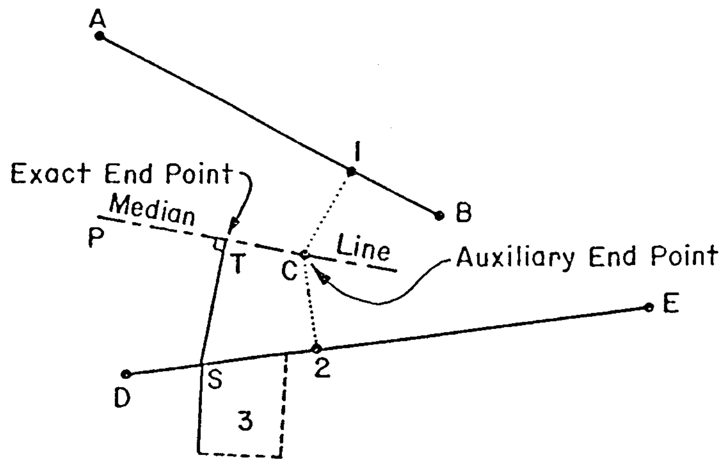


FIG. 3 — AUXILIARY END POINT AND EXACT END POINT

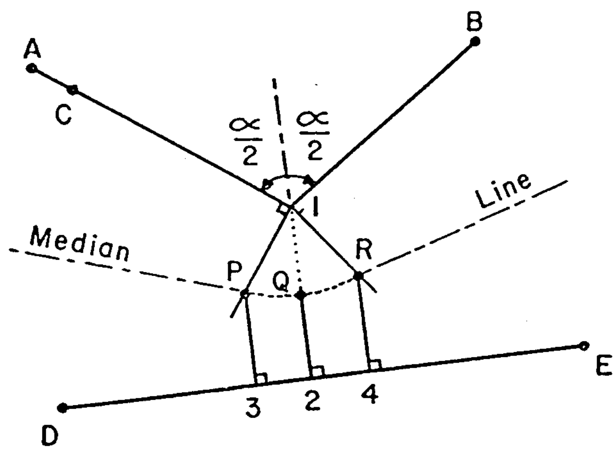


FIG. 4 — SIMPLE PARABOLIC CURVE

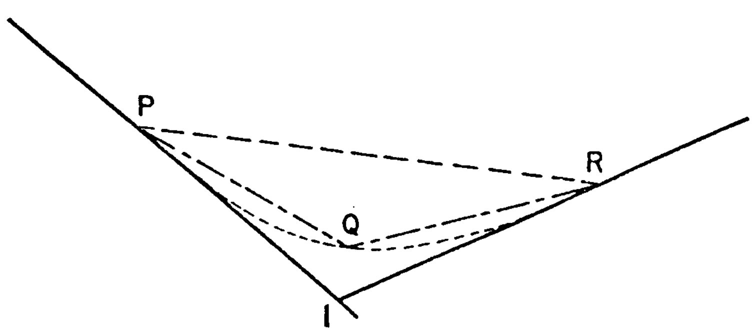


FIG. 4A — PARABOLIC CURVE APPROXIMATED BY CHORDS

LOCATION OF MEDIAN LINE AUXILIARY END POINTS

Exact points of beginning or ending of a median line are generally points of intersection that connect a median line with existing land survey lines in accordance with appropriate legal requirements. In order to compute the coordinates of exact end points, it is often helpful to locate a set of auxiliary end points on the median line which will fall outside of the exact end points, and which can be used for computing the coordinates of the exact end points.

The term auxiliary end point is applied herein to the particular type of median line end point that lies on a straight line segment of the median line, and is equidistant from (1) a straight shoreline meander segment on one shoreline and (2) a selected point on a second straight shoreline meander segment on the opposite shoreline. An end point of this type can be located in the following manner.

The approximate location of an auxiliary end point such as point C in Fig. 3 can be determined graphically by constructing a line through point 1 (on line AB) perpendicular to line AB. On this perpendicular, the center of a circle is located which will pass through point 1 and will be tangent to line DE (at point 2). The center of the circle corresponds to end point C.

The coordinates X_c, Y_c of auxiliary end point C are computed using Eqs. 1-12 of Appendix I. It is assumed that point C lies on a straight line segment of the median line at a point where the median line passes between two straight line meander segments AB and DE, the end point coordinates of which are known. Point C is equidistant from line DE and line AB at point 1. It also lies on a line through point 1 perpendicular to line AB.

If the position of an auxiliary end point is desired at a point which is equidistant from line DE and line AB, at a point B on line AB, the coordinates X_b, Y_b of point B are substituted for the coordinates X_1, Y_1 of point 1 in Eqs. 1, 2, 9 and 10.

COORDINATES OF EXACT END POINT

If the exact end point of the median line is a point T on a straight line segment, PC, of the median line, and if point T lies at the base of a perpendicular from the median line through some point S having known coordinates X_s, Y_s; the coordinates X_t, Y_t of point T can be determined by application of Eqs. 15-18 of Appendix I.

EXTENSION OF MEANDER TRAVERSE

A median line cannot be accurately located at the end of a shoreline meander traverse because the median line location is influenced by points on the meander line some distance away from the median line. For this reason, a shoreline meander must be extended beyond the survey limits so that any median line auxiliary end points will fall outside of the survey boundary. The coordinates of an exact end point intersection can then be computed properly.

SIMPLE PARABOLIC CURVED LINE SEGMENT

Median line direction changes occur gradually along simple parabolic curves when the median line passes between a meander line angle point on one shoreline (which is convex toward the median line) and a straight shoreline meander segment on the opposite shoreline. Over the length of the curve all points of the median line are equidistant from a point on one shoreline and a straight line segment on the opposite shoreline. A simple parabolic curve, PQR, is shown in Fig. 4.

Because the curved line segment is parabolic in shape, the coordinates of at least three points on the curve are needed to approximate the curve, and four points are needed to write the general equation. Coordinates of the beginning and ending points of the curve must be determined in order to establish its position. The third point should be near the center of the curve. A convenient point for computational purposes is a point on the bisector of the meander line angle point. A fourth point might be the midpoint of a perpendicular drawn from the straight shoreline meander segment DE on one shoreline through the meander line angle point 1 on the opposite shoreline.

The approximate locations of curve end points are determined graphically as follows. Referring to Fig. 4, two lines are drawn from meander line angle point 1 perpendicular to the two shoreline meander segments A1 and B1 that intersect at point 1. The center of a circle is then located on each perpendicular, which will pass through point 1 and will also be tangent to line segment DE on the opposite shoreline at point 3 and point 4. The centerpoints of the two circles correspond to curve end points P and R. Curve intermediate point Q is located in a similar manner by finding the center of a circle on the bisector of angle A1B which will pass through point 1 and will also be tangent to line DE (at point 2). The coordinates Xp, Yp and Xr, Yr of curve end points P and R determined using Eqs. 1-18 of Appendix II.

Points P and R are equidistant from point 1 and line DE. A line connecting points P and 1 would be perpendicular to the line A1 at point 1, and a line connecting points R and 1 would be perpendicular to line B1 at point 1.

The coordinates Xq, Yq of intermediate point Q, which is equidistant from point 1 and line DE, and which lies on the bisector of angle 1 are determined by application of Eqs. 21-28 of Appendix II.

If the coordinates of points P, Q and R are known, the curve can be approximated by chords. If the coordinates of a fourth point are also known; or if we know (1) the coordinates of the focal points X1, Y1 (2) the coordinates of the end points of the directrix DE and (3) the end points of the tangents to the curve at points P and R, we then have sufficient information to determine the general equation of the parabola.

If the curve is short and flat, it can sometimes be approximated by the chord PR, as shown in Fig. 4A, or it might be replaced by a single angle point, (1) at the point of intersection of the tangents. A somewhat longer curve can often be approximated by two chords PQ and QR, or a very long curve might be approximated by several chords connecting a series of points on the curve, the coordinates of which are determined using the general equation of the parabola.

A line drawn through point P perpendicular to shoreline meander segment DE intersects line DE at point 3. The coordinates of point 3 are obtained from Eqs. 13 and 14 of Appendix II.

Similarly the coordinates X4, Y4 of point 4 on line DE, which lies at the base of a perpendicular through point R, are given by Eqs. 19 and 20.

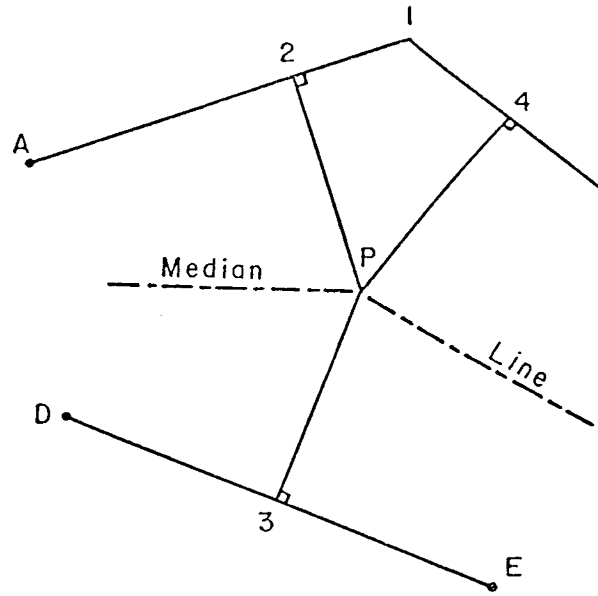


FIG. 5 — ANGLE POINT

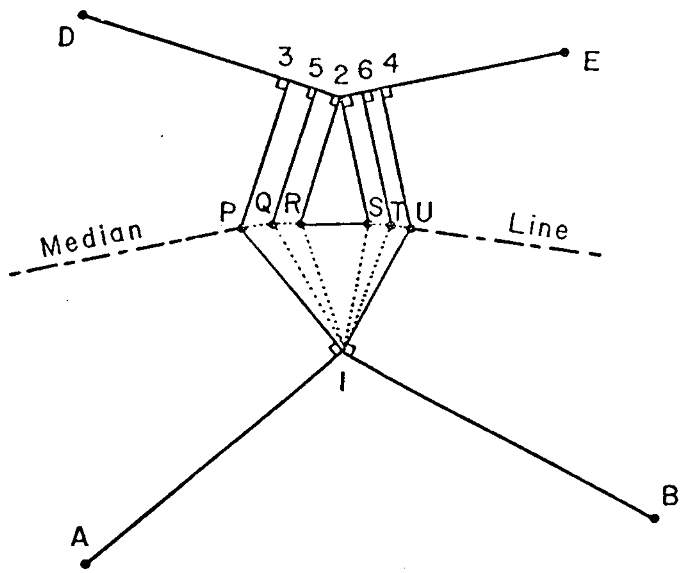


FIG. 6 — SEMI-COMPOUND CURVE

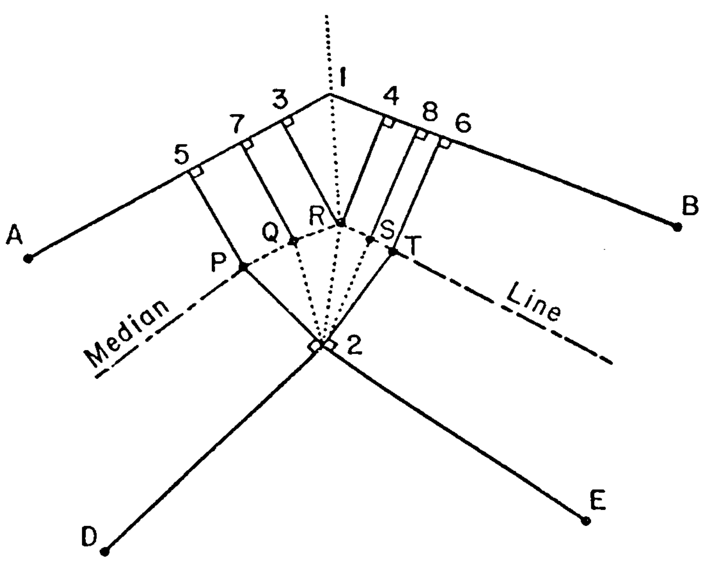


FIG. 7 — COMPOUND CURVE

MEDIAN LINE ANGLE POINT

Straight line segments of a median line exist where all points on the line are equidistant from two straight meander line segments, one on each opposite shoreline and running more or less parallel to each other.

Median line direction changes occur aburptly as angular deflections, (i.e., median line angle points) when the median line passes between a meander line angle point on one shoreline (which is concave toward the median line) and a straight shoreline meander segment on the opposite shoreline. At an angle point the median line is equidistant from three (or more) straight shoreline meander segments one of which is located on the shoreline opposite the other two.

The approximate position of an angle point can be determined graphically by locating the center of a circle that is tangent to the three shoreline meander segments as shown in Fig. 5. The coordinates of an angle point can then be obtained analytically in the manner described below.

The coordinates Xp, Yp of a median line angle point P are computed using Eqs. 1-26 of Appendix III when P is equidistant from the three straight shoreline meander segments, A1, B1 and DE, the end point coordinates Xa, Ya, Xb, Yb, X1, Y1, Xd, Yd, and Xe, Ye of which are known. As shown in Fig. 5, line segments A1 and B1 intersect at point 1, and are located on the opposite shoreline from line segment DE.

DOUBLE CURVATURE

When a median line passes between two meander line angle points located on opposite shorelines, a curving line will be formed consisting of various combinations of short parabolic curved line segments and straight line segments. If meander line angle points on opposite shorelines are adquately separated, the median line will form independent parabolic curves separated by straight line segments each of which can be treated as previously described for simple parabolic curved line segments. Frequently, however, angle points on opposite shorelines will be spaced in relation to each other in such a way that the parabolic curves associated with those points will be intersecting. The three most commonly occurring curves of this type are described following, and a fourth that occurs very rarely is included for completeness.

SEMI-COMPOUND CURVE

A median line semi-compound curve consists of two parabolic curves, curving in the same direction, and connected by a straight line segment. The straight line segment in this case is at all points equidistant from two opposite shoreline points. This characteristic distinguishes it from two independent simple curves separated by a straight line segment that is everywhere equidistant from two straight line segments of the shoreline meander lines. In addition, if the curve is semi-compound, the two opposite shoreline meander angle points will always be convex toward the median line. A curve of this type is shown in Fig. 6.

The approximate locations of semi-compound curve end points are determined graphically as follows. Referring to Fig. 6, two lines are drawn from shoreline meander angle point 1 perpendicular to the two straight shoreline meander segments A1 and B1 that intersect at point 1. The center of a circle is then located on each perpendicular which will pass through point 1 and will also be tangent to line segments D2 and E2 on the opposite shoreline at points 3 and 4. The center points of the two circles correspond to curve exterior end points P and U. Curve interior end points R and S are located in a similar manner by constructing perpendiculars to the two shoreline meander segments D2 and E2 at point 2. The center of a circle is then located on each perpendicular which will pass through point 1 and point 2.

The coordinates Xp, Yp and Xu, Yu of exterior end points P and U are determined using Eqs. 1 - 20 of Appendix IV.

The coordinates Xr, Yr and Xs, Ys of interior end points R and S are determined using Eqs. 23 - 31 of Appendix IV.

Points R and S are equidistant from two points (1 and 2) having known coordinates, and lie on lines passing through point 2 that are perpendicular to the straight shoreline meander segments D2 and E2.

Points R and S are connected by a straight line segment RS all points of which are equidistant from points 1 and 2.

The coordinates of intermediate points Q and T on median line curved line segments PR and SU, as shown in Fig. 6, are obtained by application of Eqs. 32 through 47 of Appendix IV. Points 5 and 6 are located at the midpoints of shoreline meander segments 3-2 and 2-4 repsectively.

COMPOUND CURVE

A median line compound curve consists of two parabolic curves which intersect at a point and which bend in the same direction around the same shoreline angle point. A median line compound curve is illustrated in Fig. 7.

The approximate locations of compound curve end points are determined graphically as follows. Referring to Fig. 7, two lines are drawn from shoreline meander angle point 2 perpendicular to the two shoreline meander line segments D2 and E2 that intersect at point 2. The center of a circle is then located on each perpendicular that will pass through point 2 and will also be tangent to line segments A1 and B1 on the opposite shoreline (at points 5 and 6). The center points of the two circles correspond to curve exterior end points P and T. Compound curve midpoint R corresponds to the center of a circle that is tangent to shoreline meander line segments A1 and B1 (at points 3 and 4) and which also passes through shoreline meander point 2 on the opposite shoreline.

The coordinates Xp, Yp and Xt, Yt of exterior end points P and T are determined using Eqs. 31 through 49 of Appendix V; coordinates Xr, Yr of the point of intersection R are obtained from Eqs. 1 through 20 of Appendix V; and the coordinates of points Q and S are given by Eqs. 52 through 67. Points 7 and 8 are located at the midpoints of shoreline meander segments 5-3 and 4-6 respectively.

SEMI-REVERSE CURVE

A median line semi-reverse curve consists of two parabolic curves, curving in opposite directions, that are connected by a straight line segment as shown in Fig. 8. The straight line segment, as in the case of a semi-compound curve, is at all points equidistant from two opposite shoreline meander angle points.

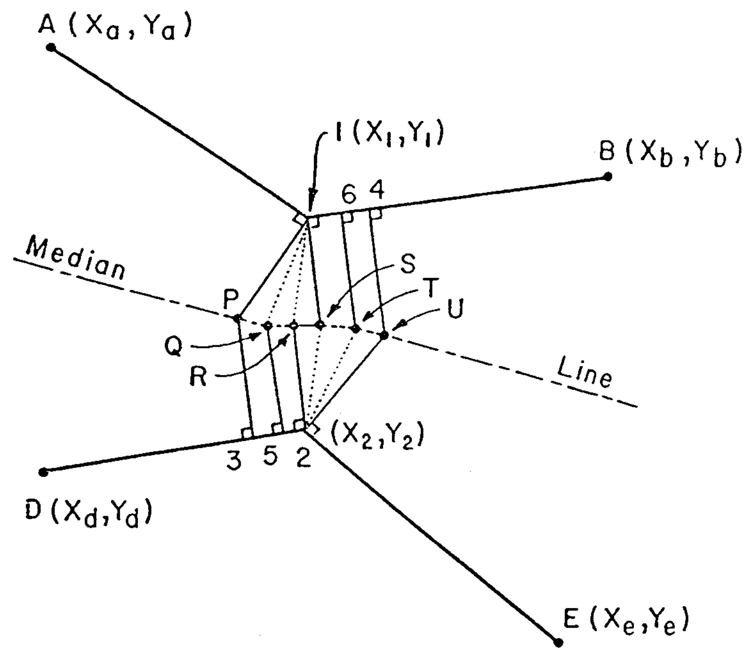


FIG. 8 – SEMI-REVERSE CURVE

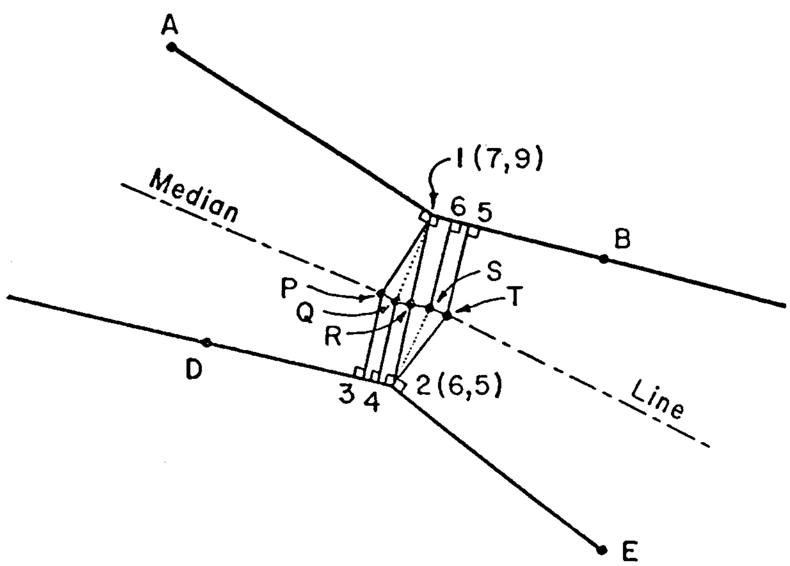


FIG. 9 – REVERSE CURVE

The approximate locations of semi-reverse curve end points are determined graphically as follows. Referring to Fig. 8, two lines are drawn from shoreline meander angle point 1 perpendicular to the two shoreline meander line segments A1 and B1 that intersect at point 1. The center of a circle is then located on each perpendicular. In the case of exterior end point P the circle will pass through point 1 and will be tangent to line D2 on the opposite shoreline at point 3. In the case of interior end point S the circle will pass through point 1 and point 2. In a similar manner lines are drawn from shoreline meander angle point 2 perpendicular to lines D2 and E2, and centers of circles are located on these two perpendiculars which, in the case of exterior end point U, will pass through point 2 and will be tangent to line B1 at point 4; and in the case of interior end point R, will pass through point 1 and point 2.

The coordinates of exterior end points P and U are computed using Eqs. 1-22 of Appendix VI, and the coordinates of interior end points R and S are computed using Eqs. 25-36 of Appendix VI.

The coordinates of intermediate points Q and T on median line curved line segments PR and SU as shown in Fig. 8, are obtained by application of Eqs. 37-52. Points 5 and 6 are located at the midpoints of shoreline meander segments 3-2 and 1-4 respectively.

REVERSE CURVE

A median line reverse curve consists of two parabolic curves which intersect at a point of tangency and which bend in opposite directions around two opposite shoreline meander angle points. A median line reverse curve is shown in Fig. 9. Curves of this type occur rarely.

The approximate locations of reverse curve end points are determined graphically as follows. Referring to Fig. 9, two lines are drawn from shoreline meander angle point 1 perpendicular to the two shoreline meander segments A1 and B1 that intersect at point 1. The center of a circle is then located on each perpendicular. In the case of exterior end point P the circle will pass throught point 1 and will be tangent to line D2 on the opposite shoreline (at point 3). In the case of interior end point R, the circle will pass throught point 1 and point 2. In a similar manner, lines are drawn from shoreline meander angle point 2 perpendicular to lines D2 and E2, and centers of circles are located on these two perpendiculars which, in the case of point T, will pass through point 2 and will be tangent to line B1 (at point 5). If the curve is truly a reverse curve the second perpendicular will also pass through the point R which has already been located.

The coordinates of the point of intersection of the two median line curves at point R are given by Eqs. 1 and 2 of Appendix VII.

The coordinates of exterior end points P and T are obtained using Eqs. 3-24 of Appendix VII, and the coordinates of intermediate points Q and S are computed by application Eqs. 27-41. Points 4 and 6 are located at the midpoints of shoreline meander segments 3-2 and 1-5 respectively.

SUMMARY

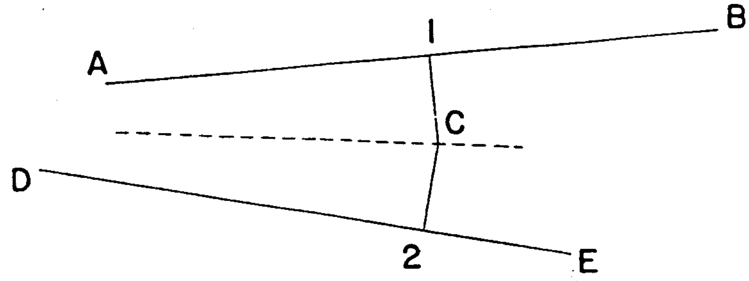
Procedures for locating geometric median lines have been outlined. Approximate locations of median line turning points are determined graphically, the geometric shoreline condition that causes the direction change is identified, and the exact coordinates of median line turning points are then computed analytically using equations presented

APPENDIX I

Auxiliary End Points

Median Line

(Sequence)
1ADE



Point C

$$M = \frac{Y_a - Y_1}{X_1 - X_a} \tag{1}$$

$$N = X_1 - MY_1 \tag{2}$$

$$E = \frac{X_d - X_e}{Y_d - Y_e} \tag{3}$$

$$F = X_e - EY_e \tag{4}$$

$$J = -\frac{1}{E} \tag{5}$$

$$P = \frac{M - J}{E - J} \tag{6}$$

$$Q = \frac{N - F}{E - J} \tag{7}$$

$$A = P[E(EP - 2M) + P - 2] \tag{8}$$

$$B = 2\{P[E(EQ + F - N) + Q] + M(X_1 - EQ - F) - Q + Y_1\} \tag{9}$$

$$C = Q[E(EQ + 2F - 2N) + Q] + F(F - 2N) + X_1(2N - X_1) - Y_1{}^2 \tag{10}$$

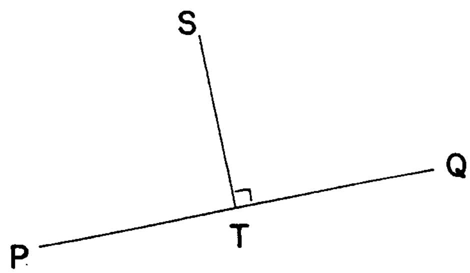
$$Y_c = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{11}$$

$$X_c = MY_c + N \tag{12}$$

Point 2 $$Y_2 = PY_c + Q \tag{13}$$

$$X_2 = EY_2 + F \tag{14}$$

Exact End Point



Point T $$M = \frac{X_q - X_p}{Y_q - Y_p} \tag{15}$$

$$N = X_p - MY_p \tag{16}$$

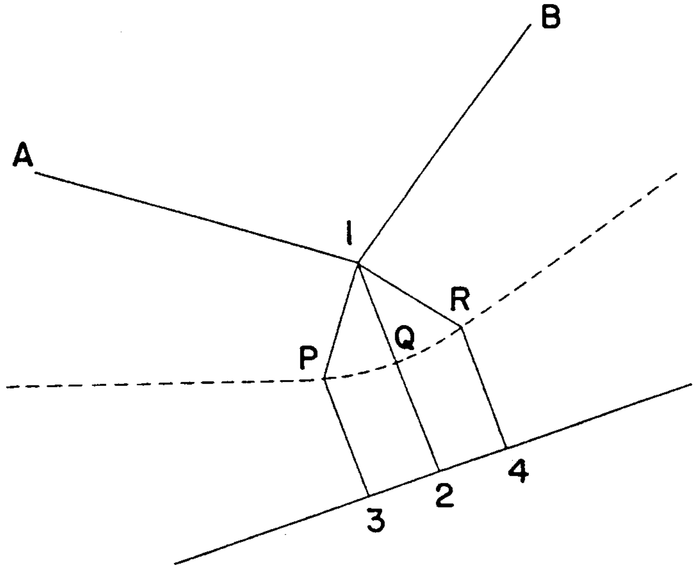
$$Y_t = \frac{Y_s + MX_s - MN}{1 + M^2} \tag{17}$$

$$X_t = MY_t + N \tag{18}$$

Median Line

APPENDIX II

Simple Parabolic Curve



AB1DE

Point P

$$M = \frac{Y_a - Y_1}{X_1 - X_a} \qquad (1)$$

$$N = X_1 - My_1 \qquad (2)$$

$$E = \frac{X_d - X_e}{Y_d - Y_e} \qquad (3)$$

$$F = X_e - Ey_e \qquad (4)$$

$$J = - \frac{1}{E} \qquad (5)$$

$$P = \frac{M - J}{E - J} \qquad (6)$$

$$Q = \frac{N - F}{E - J} \qquad (7)$$

$$A = P\left[E(EP - 2M) + P - 2\right] \tag{8}$$

$$B = 2\{P\left[E(EQ + F - N) + Q\right] + M(X_1 - EQ - F) - Q + Y_1\} \tag{9}$$

$$C = Q\left[E(EQ + 2F - 2N) + Q\right] + F(F - 2N) + X_1(2N - X_1) - Y_1^2 \tag{10}$$

$$Y_p = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{11}$$

$$X_p = MY_p + N \tag{12}$$

Point 3

$$Y_3 = PY_p + Q \tag{13}$$

$$X_3 = EY_3 + F \tag{14}$$

Point R

$$M = \frac{Y_b - Y_1}{X_1 - X_b} \tag{15}$$

$$N = X_1 - MY_1 \tag{16}$$

See Eqns. 3 through 10 (3A - 10A)

$$Y_r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{17}$$

$$X_r = MY_r + N \tag{18}$$

Point 4

$$Y_4 = PY_r + Q \tag{19}$$

$$X_4 = EY_4 + F \tag{20}$$

Point Q $$d_a = \sqrt{(X_a - X_1)^2 + (Y_a - Y_1)^2} \quad = \text{distance } \overline{A1} \tag{21}$$

$$d_b = \sqrt{(X_b - X_1)^2 + (Y_b - Y_1)^2} \quad = \text{distance } \overline{B1} \tag{22}$$

$$X_c = \frac{d_b}{d_a}(X_a - X_1) + X_1 \tag{23}$$

$$Y_c = \frac{d_b}{d_a}(Y_a - Y_1) + Y_1 \tag{24}$$

$$M = \frac{Y_b - Y_c}{X_c - X_b} \tag{25}$$

$$N = X_1 - MY_1 \tag{26}$$

See Eqns. 3 through 10 (3B - 10B)

$$Y_q = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{27}$$

$$X_q = MY_q + N \tag{28}$$

Point 2 $$Y_2 = PY_q + Q \tag{29}$$

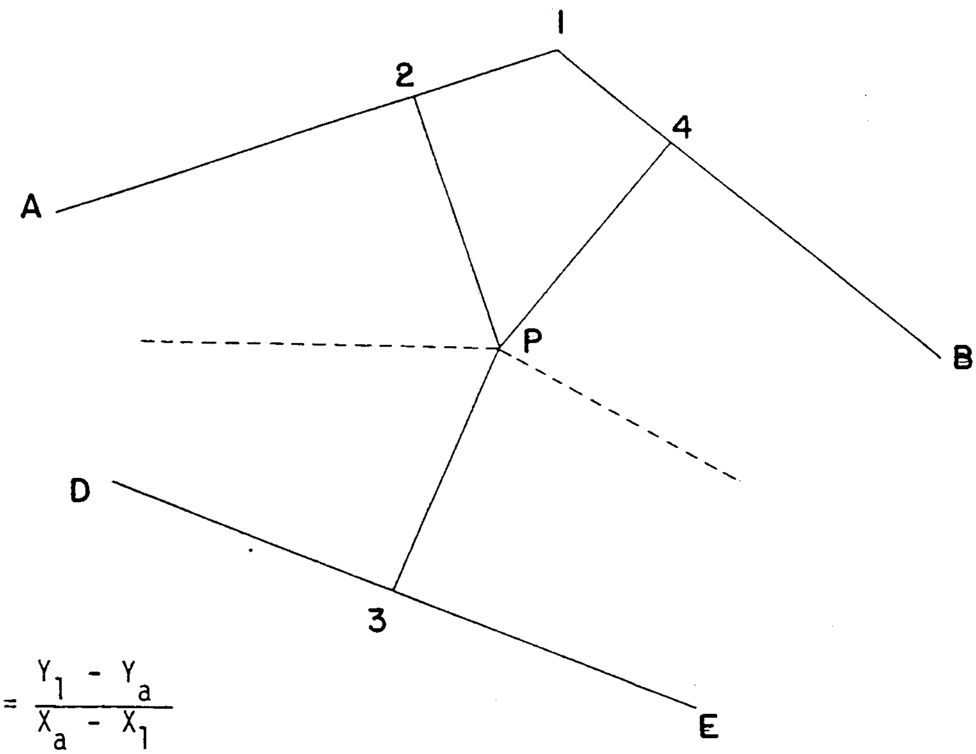
$$X_2 = EY_2 + F \tag{30}$$

Median Line

APPENDIX III

Angle Point

ABIDE



Point P

$$D = \frac{Y_1 - Y_a}{X_a - X_1} \tag{1}$$

$$E = \frac{Y_1 - Y_b}{X_b - X_1} \tag{2}$$

$$F = \frac{X_d - X_e}{Y_d - Y_e} \tag{3}$$

$$G = X_e - FY_e \tag{4}$$

$$H = \frac{X_1 - X_b}{Y_1 - Y_b} \tag{5}$$

$$J = X_b - HY_b \tag{6}$$

$$K = \frac{X_1 - X_a}{Y_1 - Y_a} = - \frac{1}{D} \tag{7}$$

$$L = X_a - KY_a \tag{8}$$

$$d_a = \sqrt{(X_a - X_1)^2 + (Y_a - Y_1)^2} \tag{9}$$

$$d_b = \sqrt{(X_b - X_1)^2 + (Y_b - Y_1)^2} \tag{10}$$

$$X_c = \frac{d_b}{d_a} (X_a - X_1) + X_1 \tag{11}$$

$$Y_c = \frac{d_b}{d_a} (Y_a - Y_1) + Y_1 \tag{12}$$

$$M = \frac{Y_b - Y_c}{X_c - X_b} \tag{13}$$

$$N = X_1 - MY_1 \tag{14}$$

$$V = \frac{Y_e - Y_d}{X_d - X_e} = - \frac{1}{F} \tag{15}$$

$$P = \frac{M - V}{F - V} \tag{16}$$

$$Q = \frac{N - G}{F - V} \tag{17}$$

$$R = \frac{M - D}{K - D} \tag{18}$$

$$S = \frac{N - L}{K - D} \tag{19}$$

$$T = \frac{M - E}{H - E} \tag{20}$$

$$U = \frac{N - J}{H - E} \tag{21}$$

$$A = P\left[F(FP - 2M) + P - 2\right] + R\left[K(2M - KR) - R + 2\right] \tag{22}$$

$$B = 2\{F[P(FQ + G - N) - MQ] + M(L - G) + S(1 - R)$$

$$+ K[R(N - KS - L) + MS] + Q(P - 1)\} \tag{23}$$

$$C = Q[F(2G + FQ - 2N) + Q] + G(G - 2N) + 2LN$$

$$+ S[K(2N - 2L - KS) - S] - L^2 \tag{24}$$

$$Y_p = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{25}$$

$$X_p = MY_p + N \tag{26}$$

Point 2 $$Y_2 = RY_p + S \tag{27}$$

$$X_2 = KY_2 + L \tag{28}$$

Point 3 $$Y_3 = PY_p + Q \tag{29}$$

$$X_3 = FY_3 + G \tag{30}$$

Point 4 $$Y_4 = TY_p + U \tag{31}$$

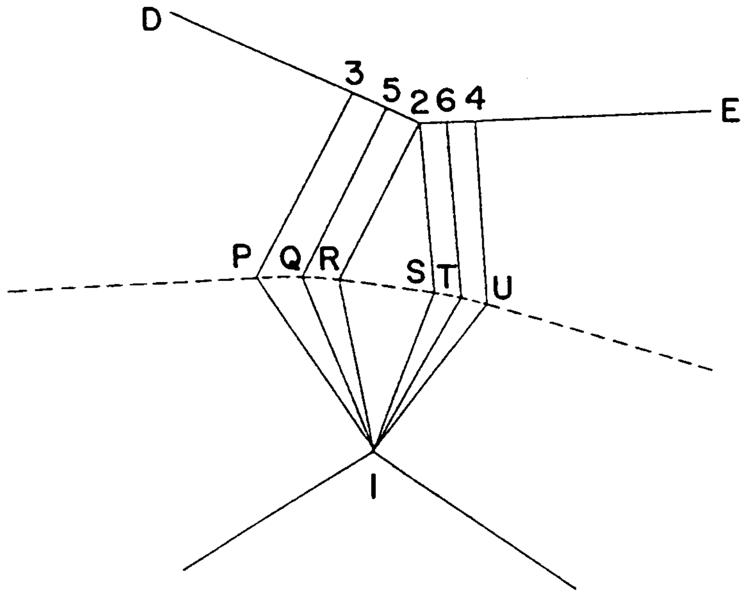
$$X_4 = HY_4 + J \tag{32}$$

Median Line

APPENDIX IV

Semi-Compound Curve

AB12DE34



Point P

$$M = \frac{Y_a - Y_1}{X_1 - X_a} \quad (1)$$

$$N = X_1 - MY_1 \quad (2)$$

$$E = \frac{X_d - X_2}{Y_d - Y_2} \quad (3)$$

$$F = X_2 - EY_2 \quad (4)$$

$$J = -\frac{1}{E} \quad (5)$$

$$P = \frac{M - J}{E - J} \quad (6)$$

$$Q = \frac{N - F}{E - J} \quad (7)$$

$$A = P\left[E(EP - 2M) + P - 2\right] \quad (8)$$

$$B = 2\{P[E(EQ + F - N) + Q] + M(X_1 - EQ - F) - Q + Y_1\} \quad (9)$$

$$C = Q[E(EQ + 2F - 2N) + Q] + F(F - 2N) + X_1(2N - X_1) - Y_1^2 \quad (10)$$

$$Y_p = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (11)$$

$$X_p = MY_p + N \quad (12)$$

Point 3 $$Y_3 = PY_p + Q \quad (13)$$

$$X_3 = EY_3 + F \quad (14)$$

Point U $$M = \frac{Y_b - Y_1}{X_1 - X_b} \quad (15)$$

$$N = X_1 - MY_1 \quad (16)$$

$$E = \frac{X_e - X_2}{Y_e - Y_2} \quad (17)$$

$$F = X_2 - EY_2 \quad (18)$$

See Eqns. 5 through 10 (5A - 10A)

$$Y_u = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (19)$$

$$X_u = MY_u + N \quad (20)$$

Point 4 $$Y_4 = PY_u + Q \quad (21)$$

$$X_4 = EY_4 + F \quad (22)$$

Point R $$J = \frac{Y_d - Y_2}{X_2 - X_d} \tag{23}$$

$$K = X_2 - JY_2 \tag{24}$$

$$G = X_2{}^2 - Y_1{}^2 - 2K(X_2 - X_1) - X_1{}^2 + Y_2{}^2 \tag{25}$$

$$H = J(X_2 - X_1) + Y_2 - Y_1 \tag{26}$$

$$Y_r = \frac{G}{2H} \tag{27}$$

$$X_r = JY_r + K \tag{28}$$

Point S $$J = \frac{Y_e - Y_2}{X_2 - X_e} \tag{29}$$

Use Eqns. 24 through 26 for evaluating G, H and K

$$Y_s = \frac{G}{2H} \tag{30}$$

$$X_s = JY_s + K \tag{31}$$

Point Q $$X_5 = \frac{X_3 + X_2}{2} \tag{32}$$

$$Y_5 = \frac{Y_3 + Y_2}{2} \tag{33}$$

$$J = \frac{Y_d - Y_5}{X_5 - X_d} \tag{34}$$

$$K = X_5 - JY_5 \quad (35)$$

$$G = X_5^2 - Y_1^2 - 2K(X_5 - X_1) - X_1^2 + Y_5^2 \quad (36)$$

$$H = J(X_5 - X_1) + Y_5 - Y_1 \quad (37)$$

$$Y_q = \frac{G}{2H} \quad (38)$$

$$X_q = JY_q + K \quad (39)$$

Point T

$$X_6 = \frac{X_4 + X_2}{2} \quad (40)$$

$$Y_6 = \frac{Y_4 + Y_2}{2} \quad (41)$$

$$J = \frac{Y_e - Y_6}{X_6 - X_e} \quad (42)$$

$$K = X_6 - JY_6 \quad (43)$$

$$G = X_6^2 - Y_1^2 - 2K(X_6 - X_1) - X_1^2 + Y_6^2 \quad (44)$$

$$H = J(X_6 - X_1) + Y_6 - Y_1 \quad (45)$$

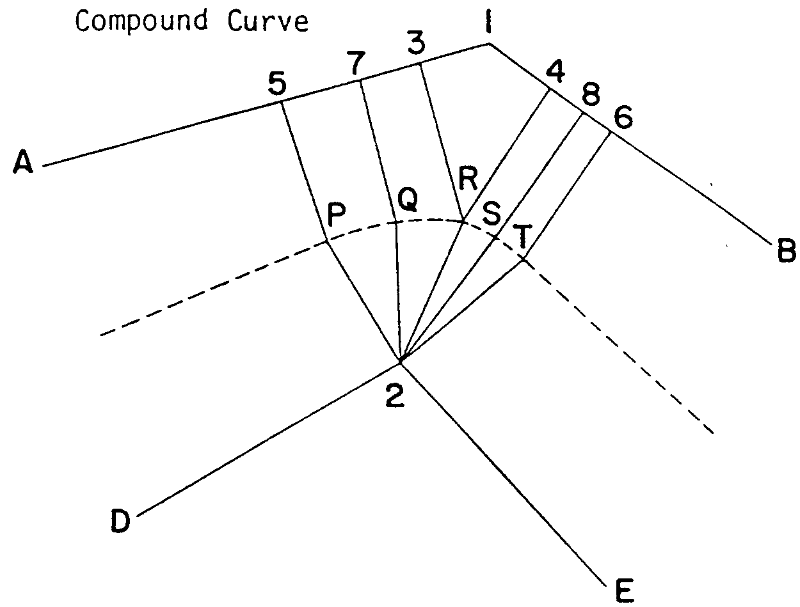
$$Y_t = \frac{G}{2H} \quad (46)$$

$$X_t = JY_t + K \quad (47)$$

Median Line

APPENDIX V

Compound Curve



AB12DE5346

Point R

$$A = \frac{X_1 - X_b}{Y_1 - Y_b} \tag{1}$$

$$B = - \frac{1}{A} \tag{2}$$

$$J = A \tag{3}$$

$$K = B \tag{4}$$

$$d_a = \sqrt{(X_a - X_1)^2 + (Y_a - Y_1)^2} \tag{5}$$

$$d_b = \sqrt{(X_b - X_1)^2 + (Y_b - Y_1)^2} \tag{6}$$

$$X_c = \frac{d_b}{d_a} (X_a - X_1) + X_1 \tag{7}$$

$$Y_c = \frac{d_b}{d_a} (Y_a - Y_1) + Y_1 \tag{8}$$

$$M = \frac{Y_b - Y_c}{X_c - X_b} \tag{9}$$

$$N = X_1 - MY_1 \tag{10}$$

$$D = X_1 - JY_1 \tag{11}$$

$$E = \frac{1}{J - K} \tag{12}$$

$$F = -KE \tag{13}$$

$$G = -DE \tag{14}$$

$$H = 1 + K^2 \tag{15}$$

$$A = H\left[EM(EM + 2F - 2) + (F - 1)^2\right] - 1 - M^2 \tag{16}$$

$$B = 2\{H\left[E(NEM + NF + GM - N) + G(F - 1)\right] + Y_2 + M(X_2 - N)\} \tag{17}$$

$$C = N(2X_2 - N + NE^2H + 2EGH) + G^2H - X_2{}^2 - Y_2{}^2 \tag{18}$$

$$Y_r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{19}$$

$$X_r = MY_r + N \tag{20}$$

Point 3

$$P = \frac{X_a - X_1}{Y_a - Y_1} \tag{21}$$

$$S = \frac{X_b - X_1}{Y_b - Y_1} \tag{22}$$

$$Q = -\frac{1}{P} \tag{23}$$

$$T = -\frac{1}{S} \tag{24}$$

$$R = X_r - QY_r \tag{25}$$

$$U = X_r - TY_r \tag{26}$$

$$Y_3 = \frac{R + PY_1 - X_1}{P - Q} \tag{27}$$

$$X_3 = QY_3 + R \tag{28}$$

Point 4

$$Y_4 = \frac{U + SY_1 - X_1}{S - T} \tag{29}$$

$$X_4 = TY_4 + U \tag{30}$$

Point P

$$M = \frac{Y_d - Y_2}{X_2 - X_d} \tag{31}$$

$$N = X_2 - MY_2 \tag{32}$$

$$E = \frac{X_a - X_1}{Y_a - Y_1} \tag{33}$$

$$F = X_1 - EY_1 \tag{34}$$

$$J = -\frac{1}{E} \tag{35}$$

$$P = \frac{M - J}{E - J} \tag{36}$$

$$Q = \frac{N - F}{E - J} \tag{37}$$

$$A = P\left[E(EP - 2M) + P - 2\right] \tag{38}$$

$$B = 2\{P[E(EQ + F - N) + Q] + M(X_2 - EQ - F) - Q + Y_2\} \quad (39)$$

$$C = Q[E(EQ + 2F - 2N) + Q] + F(F - 2N) + X_2(2N - X_2) - Y_2{}^2 \quad (40)$$

$$Y_p = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (41)$$

$$X_p = MY_p + N \quad (42)$$

Point 5 $$Y_5 = PY_p + Q \quad (43)$$

$$X_5 = EY_5 + F \quad (44)$$

Point T $$M = \frac{Y_e - Y_2}{X_2 - X_e} \quad (45)$$

$$N = X_2 - MY_2 \quad (46)$$

$$E = \frac{X_b - X_1}{Y_b - Y_1} \quad (47)$$

A, B, C, F, J, P and Q evaluated using Eqns. 34 through 40

$$Y_t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (48)$$

$$X_t = MY_t + N \quad (49)$$

Point 6 $$Y_6 = PY_t + Q \quad (50)$$

$$X_6 = EY_6 + F \quad (51)$$

Point Q $$X_7 = \frac{X_5 + X_3}{2} \tag{52}$$

$$Y_7 = \frac{Y_5 + Y_3}{2} \tag{53}$$

$$J = \frac{Y_a - Y_7}{X_7 - X_a} \tag{54}$$

$$K = X_7 - JY_7 \tag{55}$$

$$G = X_7{}^2 - Y_2{}^2 - 2K(X_7 - X_2) - X_2{}^2 + Y_7{}^2 \tag{56}$$

$$H = J(X_7 - X_2) + Y_7 - Y_2 \tag{57}$$

$$Y_q = \frac{G}{2H} \tag{58}$$

$$X_q = JY_q + K \tag{59}$$

Point S $$X_8 = \frac{X_6 + X_4}{2} \tag{60}$$

$$Y_8 = \frac{Y_6 + Y_4}{2} \tag{61}$$

$$J = \frac{Y_b - Y_8}{X_8 - X_b} \tag{62}$$

$$K = X_8 - JY_8 \tag{63}$$

$$G = X_8{}^2 - Y_2{}^2 - 2K(X_8 - X_2) - X_2{}^2 + Y_1{}^2 \tag{64}$$

$$H = J(X_8 - X_2) + Y_8 - Y_2 \tag{65}$$

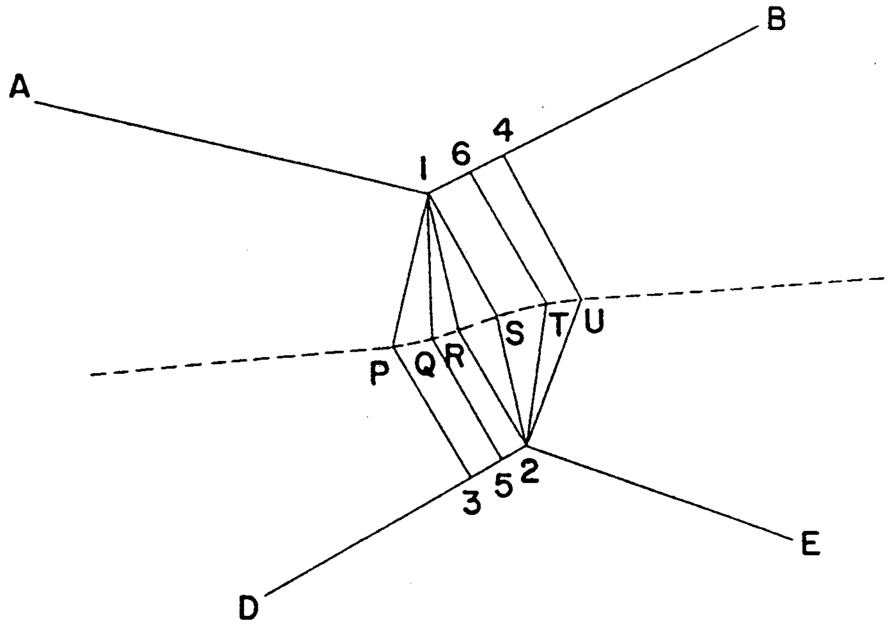
$$Y_s = \frac{G}{2H} \tag{66}$$

$$X_s = JY_s + K \tag{67}$$

Median
Line

APPENDIX VI

Semi-Reverse Curve



AB12DE34

Point P

$$M = \frac{Y_a - Y_1}{X_1 - X_a} \qquad (1)$$

$$N = X_1 - MY_1 \qquad (2)$$

$$E = \frac{X_d - X_2}{Y_d - Y_2} \qquad (3)$$

$$F = X_2 - EY_2 \qquad (4)$$

$$J = -\frac{1}{E} \qquad (5)$$

$$P = \frac{M - J}{E - J} \qquad (6)$$

$$Q = \frac{N - F}{E - J} \qquad (7)$$

$$A = P\left[E(EP - 2M) + P - 2\right] \qquad (8)$$

$$B = 2\{P[E(EQ + F - N) + Q] + M(X_1 - EQ - F) - Q + Y_1\} \tag{9}$$

$$C = Q[E(EQ + 2F - 2N) + Q] + F(F - 2N) + X_1(2N - X_1) - Y_1^2 \tag{10}$$

$$Y_p = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{11}$$

$$X_p = MY_p + N \tag{12}$$

Point 3 $$Y_3 = PY_p + Q \tag{13}$$

$$X_3 = MY_3 + N \tag{14}$$

Point U $$M = \frac{Y_e - Y_2}{X_2 - X_e} \tag{15}$$

$$N = X_2 - MY_2 \tag{16}$$

$$E = \frac{X_b - X_1}{Y_b - Y_1} \tag{17}$$

$$F = X_1 - EY_1 \tag{18}$$

A, J, P and Q are evaluated using Eqns. 5-8

$$B = 2\{P[E(EQ + F - N) + Q] + M(X_2 - EQ - F) - Q + Y_2\} \tag{19}$$

$$C = Q[E(EQ + 2F - 2N) + Q] + F(F - 2N) + X_2(2N - X_2) - Y_2^2 \tag{20}$$

$$Y_u = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{21}$$

$$X_u = MY_u + N \tag{22}$$

Point 4 $$Y_4 = PY_u + Q \tag{23}$$

$$X_4 = EY_u + F \tag{24}$$

Point R $$J = \frac{Y_d - Y_2}{X_2 - X_d} \tag{25}$$

$$K = X_2 - JY_2 \tag{26}$$

$$G = X_2{}^2 - Y_1{}^2 - 2K(X_2 - X_1) - X_1{}^2 + Y_2{}^2 \tag{27}$$

$$H = J(X_2 - X_1) + Y_2 - Y_1 \tag{28}$$

$$Y_r = \frac{G}{2H} \tag{29}$$

$$X_r = JY_r + K \tag{30}$$

Point S $$J = \frac{Y_b - Y_1}{X_1 - X_b} \tag{31}$$

$$K = X_1 - JY_1 \tag{32}$$

$$G = X_1{}^2 - Y_2{}^2 - 2K(X_1 - X_2) - X_2{}^2 + Y_1{}^2 \tag{33}$$

$$H = J(X_1 - X_2) + Y_1 - Y_2 \tag{34}$$

$$Y_s = \frac{G}{2H} \tag{35}$$

$$X_s = JY_s + K \tag{36}$$

Point Q
$$X_5 = \frac{X_3 + X_2}{2} \tag{37}$$

$$Y_5 = \frac{Y_3 + Y_2}{2} \tag{38}$$

$$J = \frac{Y_d - Y_5}{X_5 - X_d} \tag{39}$$

$$K = X_5 - JY_5 \tag{40}$$

$$G = X_5{}^2 - Y_1{}^2 - 2K(X_5 - X_1) - X_1{}^2 + Y_5{}^2 \tag{41}$$

$$H = J(X_5 - X_1) + Y_5 - Y_1 \tag{42}$$

$$Y_q = \frac{G}{2H} \tag{43}$$

$$X_q = JY_q + K \tag{44}$$

Point T
$$X_6 = \frac{X_4 + X_1}{2} \tag{45}$$

$$Y_6 = \frac{Y_4 + Y_1}{2} \tag{46}$$

$$J = \frac{Y_b - Y_6}{X_6 - X_b} \tag{47}$$

$$K = X_6 - JY_6 \tag{48}$$

$$G = X_6{}^2 - Y_2{}^2 - 2K(X_6 - X_2) - X_2{}^2 + Y_6{}^2 \tag{49}$$

$$H = J(X_6 - X_2) + Y_6 - Y_2 \tag{50}$$

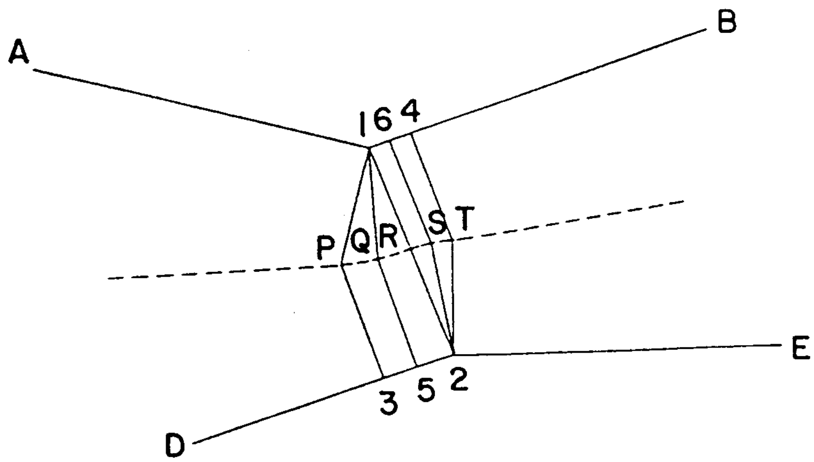
$$Y_t = \frac{G}{2H} \tag{51}$$

$$X_t = JY_t + K \tag{52}$$

APPENDIX VII

Reverse Curve

AB12DE34



Point R

$$X_r = \frac{X_1 + X_2}{2} \tag{1}$$

$$Y_r = \frac{Y_1 + Y_2}{2} \tag{2}$$

Point P

$$M = \frac{Y_a - Y_1}{X_1 - X_a} \tag{3}$$

$$N = X_1 - MY_1 \tag{4}$$

$$E = \frac{X_d - X_2}{Y_d - Y_2} \tag{5}$$

$$F = X_2 - EY_2 \tag{6}$$

$$J = -\frac{1}{E} \tag{7}$$

$$P = \frac{M - J}{E - J} \tag{8}$$

$$Q = \frac{N - F}{E - J} \tag{9}$$

$$A = P\left[E(EP - 2M) + P - 2\right] \tag{10}$$

$$B = 2\{P\left[E(EQ + F - N) + Q\right] + M(X_1 - EQ - F) - Q + Y_1\} \tag{11}$$

$$C = Q\left[E(EQ + 2F - 2N) + Q\right] + F(F - 2N) + X_1(2N - X_1) - {Y_1}^2 \tag{12}$$

$$Y_p = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{13}$$

$$X_p = MY_p + N \tag{14}$$

Point 3 $$Y_3 = PY_p + Q \tag{15}$$

$$X_3 = EY_3 + F \tag{16}$$

Point T $$M = \frac{Y_e - Y_2}{X_2 - X_e} \tag{17}$$

$$N = X_2 - MY_2 \tag{18}$$

$$E = \frac{X_b - X_1}{Y_b - Y_1} \tag{19}$$

$$F = X_1 - EY_1 \tag{20}$$

A, J, P, Q evaluated using Eqns. 7 through 10.

$$B = 2\{P\left[E(EQ + F - N) + Q\right] + M(X_2 - EQ - F) - Q + Y_2\} \tag{21}$$

$$C = Q\left[E(EQ + 2F - 2N) + Q\right] + F(F - 2N) + X_2(2N - X_2) - {Y_2}^2 \tag{22}$$

$$Y_t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{23}$$

$$X_t = MY_t + N \tag{24}$$

Point 4 $$Y_4 = PY_t + Q \tag{25}$$

$$X_4 = EY_4 + F \tag{26}$$

Point Q $$X_5 = \frac{X_3 + X_2}{2} \tag{27}$$

$$Y_5 = \frac{Y_3 + Y_2}{2} \tag{28}$$

$$J = \frac{Y_d - Y_5}{X_5 - X_d} \tag{29}$$

$$K = X_5 - JY_5 \tag{30}$$

$$G = X_5{}^2 - Y_1{}^2 - 2K(X_5 - X_1) - X_1{}^2 + Y_5{}^2 \tag{31}$$

$$Y_q = \frac{G}{2H} \tag{32}$$

$$X_q = JY_q + K \tag{33}$$

Point S $$X_6 = \frac{X_4 + X_1}{2} \tag{34}$$

$$Y_6 = \frac{Y_4 + Y_1}{2} \tag{35}$$

$$J = \frac{Y_b - Y_6}{X_6 - X_b} \tag{36}$$

$$K = X_6 - JY_6 \quad (37)$$

$$G = X_6{}^2 - Y_2{}^2 - 2K(X_6 - X_2) - X_2{}^2 + Y_6{}^2 \quad (38)$$

$$H = J(X_6 - X_2) + Y_6 - Y_2 \quad (39)$$

$$Y_s = \frac{G}{2H} \quad (40)$$

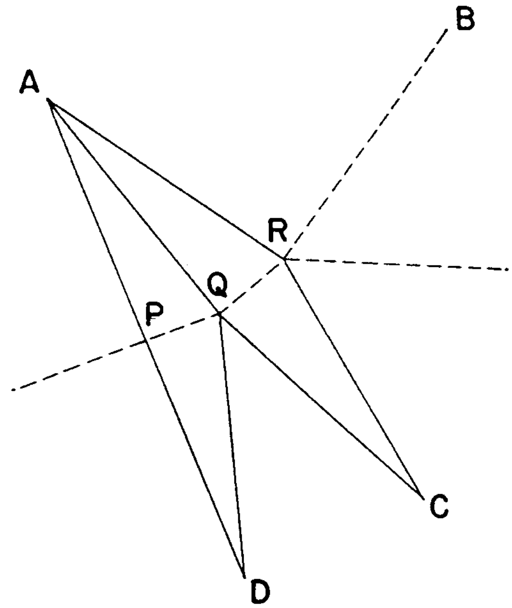
$$X_s = JY_s + K \quad (41)$$

Median Line

APPENDIX VIII

Salient Point

ABC



Point R

$$E = (Y_c - Y_a)(X_b - X_a) - (Y_b - Y_a)(X_c - X_a)$$

$$F = X_b^2 + Y_b^2 - X_a^2 - Y_a^2$$

$$G = X_c^2 + Y_c^2 - X_a^2 - Y_a^2$$

$$Y_r = \frac{G(X_b - X_a) - F(X_c - X_a)}{2E}$$

$$X_r = \frac{F - 2Y_r(Y_b - Y_a)}{2(X_b - X_a)}$$

Point P

$$X_p = \frac{X_a + X_d}{2}$$

$$Y_p = \frac{Y_a + Y_d}{2}$$